The Grothendieck group of $GL_n$

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The idea of this project is to give an alternative and easier proof to theorem 4 of [2, p. 49] in the case where $G = GL_n$ and $k \subset \mathbb{C}$ a field.

1 Semisimplicity

In this section we want to prove the semisimplicity of $GL_n$-modules over $k$. Let $0 \to \rho' \to \rho \to \rho'' \to 0$ be an exact sequence of representations over the algebraic group $GL_{n,k}$ over a field $k$ of characteristic 0. Our goal can be reformulated as follows.

Theorem 1. The functor $\text{Hom}(\rho'', -)$ from the category of representations of $GL_n$ over $k$ to the category of $k$-modules is exact.

Actually, this theorem would yield that 

$$0 \to \text{Hom}(\rho'', \rho') \to \text{Hom}(\rho'', \rho) \to \text{Hom}(\rho'', \rho'') \to 0$$

is exact and hence that the identity $\rho'' \to \rho''$ is the image of an element of $s \in \text{Hom}(\rho'', \rho)$. This element is a section of the original exact sequence.

1.1 Compatibility with extension of scalars

In this section we will prove that theorem 1 is compatible with extension of scalars in the following sense.

Lemma 2. Let $k_1 \subset k_2$ be two fields of characteristic 0. Then theorem 1 holds for $k = k_1$ if it holds for $k = k_2$.

Proof. There is a natural map $Z : \text{Hom}_{GL_n,k_1}(V, W) \otimes k_2 \to \text{Hom}_{GL_n,k_2}(V \otimes k_2, W \otimes k_2)$ and this map is injective as $\text{Hom}_{GL_n,k_i}(V, W) \subset \text{Hom}_{k_i}(V, W)$ (for $i = 1, 2$) and $Z$ is the restriction of the isomorphism $\text{Hom}_{k_1}(V, W) \otimes k_2 \cong \text{Hom}_{k_2}(V \otimes k_2, W \otimes k_2)$. Next we will proof that $Z$ is surjective.

Let $\phi \in \text{Hom}_{GL_n,k_2}(V \otimes k_2, W \otimes k_2)$. We can consider $\phi$ as a matrix and let $S \subset k_2$ be the $k_1$-vector space the matrix’ coefficients generate. It is
finite dimensional. Let $e_1, \ldots, e_j$ be a basis. As the action of $GL_n, k_1 \subset GL_n, k_2$ acts $k_1$-linear, the $k_1 e_i$-component $\phi_i$ of the map $\phi|_V$ is a morphism of $GL_n, k_1$-modules. Furthermore, $\phi_i$ is of the form $Z(\psi_i \otimes e_i)$ where $\psi_i \in \text{Hom}_{GL_n, k_1}(V, W)$. As $\phi = Z(\sum_i \psi_i \otimes e_i)$, we have proven the surjectivity now.

As $- \otimes k_2$ is an exact functor the statement immediately follows.

Remark 3. To prove the statement for fields of characteristic 0 not contained in $\mathbb{C}$ we notice that a statement like this lemma holds for inductive limits and that every field of characteristic 0 is an inductive limit of subfields of $\mathbb{C}$.

1.2 Proof for $k = \mathbb{C}$

A representation $\rho$ of $GL_n$ over $k = \mathbb{C}$ induces a representation $V$ of the group $GL_n(\mathbb{C})$ where $GL_n(\mathbb{C})$ has the usual topology. As $GL_n \rightarrow \text{Aut}_V$ is a morphism of varieties, the induced representation is smooth. We restrict this representation to the group $U_n \subset GL_n$ of unitary matrices, call it $V$. In the same way $\rho'$ and $\rho''$ induce representations $V'$ and $V''$ of $U_n$. Now we will use the following fact to prove that the sequence $0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$ splits.

Fact 4. Every locally compact Hausdorff topological group has a Haar measure.

As $U_n$ is a locally compact Hausdorff topological group we can and will equip it with a Haar measure and as $U_n$ is abelian, this measure will be both right- and left-invariant. Furthermore we may and do suppose that the measure of the whole group $U_n$ is 1 as $U_n$ is compact.

Equip $V$ with an arbitrary inner product $\langle \cdot, \cdot \rangle$. Then consider the map

$$B : V \times V \rightarrow \mathbb{C} : (v_1, v_2) \mapsto \int_{U_n} \langle gv_1, gv_2 \rangle dg.$$ 

Proposition 5. The map $B$ is an inner product of $V$ that is $U_n$-invariant.

Proof. Notice that $B(v, v) = \int \langle gv, gv \rangle dg$ is the integral of a non-negative function and hence it is non-negative. We also deduce immediately that $B(v, v) = 0$ if and only if $v = 0$. Furthermore, $B$ is clearly linear in the first argument as $\langle \cdot, \cdot \rangle$ is linear in the first argument and in the same way we have $B(v_2, v_1) = B(v_1, v_2)$. Hence, $B$ is an inner product.
Furthermore,
\[ B(v_1, v_2) = \int \langle gv_1, gv_2 \rangle dg = \int \langle gg_3v_1, gg_3v_2 \rangle dg = B(g_3v_1, g_3v_2), \]
as the Haar measure is \( U_n \)-invariant.

Let \( W \) be a space orthogonal to \( V' \) in \( V \) with respect to the inner product \( B \). Then for all \( g \in U_n \), \( w \in W \) and \( v \in V' \) we have \( B(gw, v) = B(w, g^{-1}v) = 0 \) as \( g^{-1}v \in V' \) and \( w \in W \). Hence we have \( gw \in W \) and we deduce that \( W \) is not only a subspace but in fact a subrepresentation \( \rho_W \) of \( V \).

This yields an exact sequence of representations \( 0 \to V' \to V \to W \to 0 \). In particular, \( W \) is isomorphic to \( V'' \). Finally, because \( W \subset V \), this gives us a way to split the exact sequence as we wanted to do.

The subspace \( W \) induces a subspace of \( \rho \) complement to \( \rho' \) and isomorphic to \( \rho'' \). Hence \( \rho'' \) is fixed by the subgroup \( U_n(\mathbb{C}) \subset GL_n(\mathbb{C}) \). By proposition 12.1 of [1, p. 130] the stabilizer of \( \rho'' \) is a (Zariski closed) subgroup of \( GL_n \). The following theorem will prove that \( \rho'' \) is in fact \( GL_n \)-invariant and concludes the proof that the exact sequence splits.

**Lemma 6.** The subset \( U_n(\mathbb{C}) \subset GL_n(\mathbb{C}) \subset GL_{n,\mathbb{C}} \) is Zariski dense.

**Proof.** We will prove that \( U_n(\mathbb{C}) \) is dense in \( GL_n(\mathbb{C}) \) which is dense in \( GL_{n,\mathbb{C}} \).

Let \( f \) be a polynomial on \( GL_n(\mathbb{C}) \) that is zero on \( U_n(\mathbb{C}) \). We will prove that \( f \) is the zero polynomial. Consider the map \( exp : \text{Mat}_n(\mathbb{C}) \to GL_n(\mathbb{C}) \) that exponentiates a matrix. It is known to be a surjective analytic function. In particular the function \( g = f \circ \text{exp} \) is analytic. We will prove that it is the zero function, which by the surjective of \( \text{exp} \) also proves that \( f = 0 \).

Suppose that \( M \in \text{Mat}_n(\mathbb{C}) \) is such that \( M = -M^* \). Then
\[
(exp M)^n = \sum_{n=0}^{\infty} \frac{1}{n!}(M^*)^n = \sum_{n=0}^{\infty} \frac{1}{n}(-M)^n = \exp(-M) = \exp(M)^{-1}.
\]
Hence, \( \exp(M) \in U_n(\mathbb{C}) \) and \( g(M) = 0 \) for all \( M \in \text{Mat}_n(\mathbb{C}) \) such that \( M = -M^* \). For \( i, j \in \{1, \ldots, n\} \) let \( E_{ij} \) be the matrix with a 1 in the \((i, j)\)-th entry and zeros elsewhere. For \( i \in \{1, \ldots, n\} \) let \( A_i = i \cdot E_{ii} \). For \( 1 \leq i < j \leq n \) let \( B_{ij} = E_{ij} - E_{ji} \) and let \( C_{ij} = iE_{ij} + iE_{ji} \). Then the \( A_i \), \( B_{ij} \) and \( C_{ij} \) together form a \( \mathbb{C} \)-basis of the vector space \( \text{Mat}_n(\mathbb{C}) \). Moreover the basis vectors satisfy \( M = -M^* \).
In other words, we can identify Mat$_n$(C) with C$^{n^2}$ in such a way that in this identification we have $g|_{R^{n^2}} = 0$. By the theory of complex analysis now follows that all partial derivatives of $g$ in the point $0 \in C^{n^2}$ must be 0 and as $g$ is analytic this yields that $g = 0$ on the whole $C^{n^2} \cong$ Mat$_n$(C). \qed
2 Character theory

Let $\rho$ be a representation of $\text{GL}_n$ over $k$. Then $\rho$ induces a representation of $\text{GL}_n(k)$ and we define its character to be the function $\chi_\rho : \text{GL}_n(k) \to k : g \mapsto \text{Tr}(\rho(g))$. The goal of this chapter is to prove the following theorem.

**Theorem 7.** Two finite-dimensional representations $\rho_1$ and $\rho_2$ of $\text{GL}_n$ over $k$ are isomorphic if and only if their characters are equal.

2.1 Preliminary results

The following results will be needed to prove theorem 7.

**Lemma 8.** Let $A$ be a ring and $E$ be a simple $A$-module. Let $N$ be the Jacobsen radical of $A$. Then $NE = 0$.

**Proof.** As $E$ is simple it is generated by one element, say $e \in E$. Let $I = \text{Ann}(e) := \{a \in A : ae = 0\}$; it is a left ideal of $A$. Then $E \cong A/I$. The submodules of $E$ correspond to the left ideals of the ring $A$ containing $I$. As $E$ is simple, we deduce that there are two such ideals and hence that $I$ is a maximal left ideal of $A$. But then we have $I \supset N$ and hence $NE = 0$. □

**Corollary 9.** Let $A$ be a ring and $E$ be a semisimple $A$-module. Let $N$ be the Jacobsen radical of $A$. Then $NE = 0$.

**Theorem 10** (Artin-Wedderburn). Let $A$ be a commutative ring. Suppose that $A$ is artinian and that its Jacobsen radical is zero. Then $A$ is a finite product of matrix rings over division rings. □

2.2 Proof of the theorem

**Proof.** Let $V_1$ and $V_2$ be two $k[\text{GL}_n(k)]$-modules that are finite-dimensional as $k$-vector space and have the same character. By the results from the first chapter, we know that $V_1$ and $V_2$ are semisimple. Let $N$ be the kernel of the natural map $\text{GL}_n(k) \to \text{End}_k(V_1 \oplus V_2)$ and let $B = \text{GL}_n(k)/N$. Then $V_1$ and $V_2$ are $B$-modules and $B$ acts faithfully on $V_1 \oplus V_2$, hence $B$ is finite dimensional as $k$-vector space.

Of course $V_1$ and $V_2$ are semisimple $B$-modules, as their simple components remain simple over $B$. Hence, by corollary 9 the Jacobsen radical of $B$ acts
trivially on both $V_1$ and $V_2$ and hence it is 0. As $B$ is finite dimensional over $k$ it is certainly artinian. Hence by the Artin-Wedderburn theorem, we have

$$B = \text{Mat}(D_1, n_1) \times \cdots \times \text{Mat}(D_s, n_s),$$

where for $i = 1, \ldots, s$ we have $n_i \in \mathbb{Z}_{>0}$ and $D_i$ is a finite dimensional division algebra over $k$.

Notice that as a $B$-module $\text{Mat}(D_i, n_i)$ is isomorphic to the product of $n_i$ copies of the simple module $D_i^{n_i}$, where $B$ acts in the obvious way (the $i$-th factor acts by multiplication and the other factors by zero). In particular we deduce that the simple modules are isomorphic to the $D_i^{n_i}$.

Let $\pi_i \in B$ be such that $\pi_i|_{\text{Mat}(D_i, n_i)} = 1$ and $\pi_i|_{\text{Mat}(D_j, n_j)} = 0$ for all $i, j \in \{1, \ldots, s\}$ such that $i \neq j$. Then $\chi_{V_1}(\pi_i)$ is the number of factors $D_i^{n_i}$ in $V_1$. The same holds for $V_2$. Together with the fact that the characteristic of $k$ is 0 this proves that $V_1$ and $V_2$ are isomorphic.

For two $\text{GL}_{n,k}$-modules with the same character, their underlying $\text{GL}_n(k)$-modules are isomorphic. This gives a isomorphism of vector spaces $V_1 \to V_2$ that commutes with the action of $\text{GL}_n(k)$. As $\text{GL}_n(k) \subset \text{GL}_{n,k}$ is Zariski dense, the isomorphism in facts commutes with $\text{GL}_{n,k}$ and is a isomorphism of $\text{GL}_{n,k}$-modules. 

\[\square\]
3 Main statement

In the last chapter we will finally prove theorem 4 of [2, p. 49].

By the corollary of proposition 7 of [2, p. 48] the Grothendieck group of the subgroup $D \subset \text{GL}_n$ of diagonal matrices is isomorphic to the group $H := \mathbb{Z}[X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}]$. This isomorphism is called $\text{ch} : R_k(D) \to H$. If $V$ is a $D$-comodule, then the $X_1^{i_1} \cdots X_n^{i_n}$ coefficient of $\text{ch}(V)$ is the rank of $\{v \in V : dv = X_1^{i_1} \cdots X_n^{i_n} \otimes v\}$.

If we compose $\text{ch}$ with the restriction $R_k(\text{GL}_n) \to R_k(D)$ we obtain a map that is called $\text{ch}_G : R_k(\text{GL}_n) \to H$.

**Theorem 11.** The homomorphism $\text{ch}_G$ is injective. Its image is the subgroup $H^W$ of $H := \mathbb{Z}[X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}]$ formed by the elements that are invariant under $W = S_n$, where $W$ acts on $H$ by permutation of the $X_i$.

3.1 Injectivity

We will factor the map $\text{ch}_G$ via the character group $X := \{\chi_V : \text{GL}_n(k) \to k : V$ is a $G$-representation\}. In the previous section we have proved that the map from $R_k(\text{GL}_n) \to X$ is injective. Now we will consider why the map $X \to H$ is injective, proving that the composition is injective.

Proposition 7 of [2, p. 48] tells us that with each element of $f \in H$ corresponds a comodule, say $E$. For each monomial $m \in H$ the module $E$ has an $m$-component of rank equal to the coefficient of $m$ in $f$, $f_m$. We have $E = \bigoplus E_m$.

Let $D \subset \text{GL}_n$ be the (diagonalizable) subgroup of diagonal matrices and let $M \in D(k)$ be an arbitrary element. Write $M = \text{diag}(d_1, \ldots, d_n)$, then $M$ acts on the $m$-component by multiplication with $f_m \cdot m(d_1, \ldots, d_n)$. In particular $\chi_E(M) = f(d_1, \ldots, d_n)$. The fact that $X \to H$ is injective follows from the fact that there is only one polynomial when we fix a set of values in all points of $(\mathbb{Z} \setminus 0)^n$.

3.2 Image

We will prove our result by proving the following two lemmas.

**Lemma 12.** The image of $\text{ch}_G$ is contained in $H^W$. 

Proof. As $k$ is commutative, we have $\chi(AB) = \chi(BA)$ for all $A, B \in \text{GL}_n(k)$. Let $\sigma \in S_n$ and consider the matrix $P$ that permutes the standard basis by $\sigma$. Then for all $M \in D(k)$ as in the previous section, we have $\chi(PMP^{-1}) = \chi(M)$. In particular, in the terms of the proof in the last section, we must have $f(d_1, \ldots, d_n) = f(\sigma(d_1, \ldots, d_n))$. Hence, the polynomial $f \circ \sigma$ must be equal to the polynomial $f$ for all $\sigma \in S_n$ and hence $f \in H^{S_n}$.

Lemma 13. The subset $H^{S_n}$ is contained in the image of $\text{ch}_G$.

Proof. Let $V = k^n$ and let $\text{GL}_n(k)$ act on it by multiplication. It naturally extends to a $\text{GL}_{n,k}$-module. Clearly $M = (d_1, \ldots, d_n) \in D(k)$ acts on the basis vectors $e_i$ of $V$ by multiplication with $d_i$. Hence, $\chi_V(d_1, \ldots, d_n) = d_1 + \ldots + d_n$ and $X_1 + \ldots + X_n$ is in the image of $\text{ch}_G$.

For the other symmetric polynomials $s_i$ of degree $i$ we consider $\wedge^i V$. The basis vectors are of the form $e_{j_1} \wedge \ldots \wedge e_{j_i}$ and $M$ acts on it by multiplication with $d_{j_1} \ldots d_{j_i}$. This proves that $\chi_V(d_1, \ldots, d_n) = s_i(d_1, \ldots, d_n)$ and hence $s_i$ is in the image of $\text{ch}_G$.

As a ring $H^{S_n}$ is generated by the $s_i$. As the character of a tensor product of representations is the product of characters, the lemma can now be considered to be proven.

References
