Jordan decomposition and Tannaka duality

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These informal talk notes are mostly due to [1] and are prone to errors. I can also recommend sections 2.4 and 2.5 of [2].

1 Jordan decomposition

1.1 Case of finite dimensional vector spaces

Let \( k \) be a perfect field, let \( V \) be a finite dimensional \( k \)-vector space and let \( \alpha : V \to V \) be a \( k \)-automorphism. Then \( \alpha \) is called diagonalizable if \( V \) has a basis of eigenvectors, \( \alpha \) is called semisimple if \( \alpha \otimes K \) is diagonalizable for some field extension \( K/k \), \( \alpha \) is called nilpotent if \( \alpha^m = 0 \) for some \( m \in \mathbb{Z}_{\geq 0} \), and \( \alpha \) is called unipotent if \( \alpha - 1 \) is nilpotent. Let \( E = E(\alpha) \) be the set of eigenvalues of \( \alpha \) in \( k \) and for \( a \in E \) let \( V_a := \{ v \in V : \exists N : (\alpha - a)^N v = 0 \} \) be the associated generalized eigenspace.

**Proposition 1.** If all eigenvalues of \( \alpha \) lie in \( k \), then

\[
V = \bigoplus_{a \in E} V^a.
\]

**Proof.** Omitted, see proposition 2.1 of [1, p. 155]. □

**Theorem 2** (Jordan decomposition). There exist unique \( k \)-automorphisms \( \alpha_s, \alpha_u : V \to V \) such that \( \alpha_s \) is semisimple, \( \alpha_u \) is unipotent, and \( \alpha = \alpha_s \circ \alpha_u = \alpha_u \circ \alpha_s \).

**Proof.** First we prove the uniqueness. Suppose that \( \alpha_s \circ \alpha_u \) and \( \beta_s \circ \beta_u \) are two such decomposition, then \( \beta_s^{-1} \circ \alpha_s = \beta_u \circ \alpha_u^{-1} \) is semisimple and nilpotent, hence it is equal to the identity map. This proves the uniqueness.

For the existence first consider the case where all eigenvalues are in \( k \). By proposition 1 we have \( V = \bigoplus_{a \in E} V^a \). Let \( \alpha_s : V \to V \) be such that \( \alpha|_{V^a} \) is the multiplication by \( a \). Now, let \( \alpha_u := \alpha \circ \alpha_s^{-1} \). Then \( \alpha_s \) is semisimple by construction, \( \alpha_u \) is unipotent because all its eigenvalues are 1 and \( \alpha_s \) and \( \alpha_u \) commute. This proves the existence when all eigenvalues are in \( k \).
In the general case the eigenvalues lie in a finite field extension $K/k$. Because $k$ is perfect, we may and do assume that $K/k$ is finite Galois with Galois group $G$. Let $\alpha_s \circ \alpha_u$ be the Jordan decomposition of $\alpha \otimes K$. Then it is easy to check that $(\sigma \alpha_s) \circ (\sigma \alpha_u)$ is also a Jordan decomposition for all $\sigma \in G$. Hence, $\sigma \alpha_s = \alpha_s$ and $\sigma \alpha_u = \alpha_u$. Hence $\alpha_s$ and $\alpha_u$ are defined over $k$ and the Jordan decomposition of $\alpha$ is $\alpha_s|_V \circ \alpha_u|_V$.

**Lemma 3.** Let $\alpha$ and $\beta$ be $k$-automorphisms of finite dimensional $k$-vector spaces $V$ and $W$. Let $\phi : V \to W$ be a $k$-morphism. Suppose that $\phi \circ \alpha = \beta \circ \phi$. Then we have $\phi \circ \alpha_s = \beta_s \circ \phi$ and $\phi \circ \alpha_u = \beta_u \circ \phi$.

*Proof.* It suffices to prove this in the case where all eigenvalues are in $k$. Let $a \in E(\alpha)$. Then it is easy to check that $\phi(V^a) \subset W^a$. Hence, on $V^a$ the maps $\phi \circ \alpha_s$ and $\beta_s \circ \phi$ agree. The same is true for $\phi \circ \alpha_s^{-1}$ and $\beta_s^{-1} \circ \phi$. Hence by proposition 1 the maps agree on $V$.

**Corollary 4.** Let $W$ be a subspace of $V$, then $\alpha|_W = \alpha_s|_W \circ \alpha_u|_W$ is the Jordan decomposition of $\alpha|_W$.

**Lemma 5.** Let $\alpha$ and $\beta$ be $k$-automorphisms of finite dimensional $k$-vector spaces $V$ and $W$. Then $(\alpha \otimes \beta)_s = \alpha_s \otimes \beta_s$ and $(\alpha \otimes \beta)_u = \alpha_u \otimes \beta_u$.

*Proof.* Similar to the proof of lemma 3, see proposition 2.5 of [1, p. 157].

### 1.2 Case of infinite dimensional vector spaces

Let $k$ be a perfect field and $V$ an arbitrary $k$-vector space. A $k$-automorphism $\alpha : V \to V$ is called *locally finite* if $V$ is a union of finite dimensional $\alpha$-stable subspaces. The notions of a semisimple, nilpotent and unipotent automorphism extend.

**Theorem 6** (Jordan decomposition). *Theorem 2 also holds for arbitrary $V$ and locally finite $\alpha$.*

*Proof.* Every $\alpha$-stable subspace has a unique Jordan decomposition and these coincide by corollary 4.
1.3 Case of algebraic groups

Theorem 7 (Jordan decomposition in algebraic groups). Let $G$ be an affine algebraic group over a perfect field $k$. For any $g \in G(k)$ there are unique $g_s, g_u \in G(k)$ such that for all (locally finite) representations $r : G \to \text{Aut}(V)$ we have $r(g_s) = r(g)_s$ and $r(g_u) = r(g)_u$. Furthermore $g_s g_u = g_u g_s = g$.

Proof. The theorem follows from Tannaka duality applied to the family $(r(g)_s)_r$ and $(r(g)_u)_r$ where $r$ ranges over all finite dimensional representations. By choosing a faithful representation $r$ we find $r(g) = r(g_s)r(g_u) = r(g_u)r(g_s)$ and hence the desired equality.

2 Tannaka duality

Let $G$ be an affine algebraic group over a field $k$ (in fact we can do this over a noetherian ring $k$) with coordinate ring $A$ and let $R$ be a $k$-algebra.

2.1 Statement

Tannaka duality allows us to reconstruct the group $G$ when we only have some limited knowledge about its representations.

Theorem 8 (Tannaka duality). Suppose that for every representation $r_V : G \to \text{Aut}(V)$ which is finitely generated as $k$-module we have an $\alpha_V : V_R \to V_R$ such that

(a) if $V$ and $W$ are representations, then $\alpha_V \otimes W = \alpha_V \otimes \alpha_W$;
(b) if $\phi : V \to W$ is a homomorphism of $G$-modules then $\phi_R \circ \alpha_V = \alpha_W \circ \phi_R$;
(c) $\alpha_k = 1$.

Then there exists a unique $g \in G(R)$ such that $\alpha_V = r_V(g)$ for every $V$.

Proof of theorem 7. The conditions (a), (b) and (c) are satisfied because of lemmas 5 and 3.
2.2 Some lemmas

Let \( \Delta : A \to A \otimes A \) be the comultiplication. Furthermore let \( r_A : G \to \text{End}_A \) be the regular representation, i.e. for every \( k \)-algebra \( R \) we let \( g \in G(R) \) act on \( f \in A \) by

\[
\forall x \in G(R) : (gf)_R(x) = f_R(x \cdot g),
\]

where we consider \( f \in A \) as regular function \( G \to k \). To prove theorem 8 we need the following lemmas.

**Lemma 9.** Let \( u : A \to A \) be a \( k \)-algebra endomorphism such that \( \Delta \circ u = (1 \otimes u) \circ \Delta \). Then there exists a \( g \in G(k) \) such that \( u = r_A(g) \).

**Proof.** Let \( \phi : G \to G \) be the morphism corresponding to \( u \). Let \( m : G \times G \to G \) be the multiplication (corresponding to \( \Delta \)). Then we have

\[
\phi_R(x \cdot y) = \phi_R(m_R(x, y)) = m_R(x, \phi_R(y)) = x \cdot \phi_R(y).
\]

By choosing \( y = e \) in (1) we find \( \phi_R(x) = x \cdot g \) where \( g = \phi_R(e) \). Then the correspondence yields us that \( u = r_A(g) \).

**Lemma 10.** Every representation \( V \) of \( G \) is a union of its finitely generated subrepresentations, or otherwise stated representations of \( G \) are locally finite.

**Proof.** Already given on 19 February, see proposition 6.6 in [1, p. 121].

**Lemma 11.** Let \( r_V : G \to \text{Aut}(V) \) be a representation of \( G \) finitely generated as \( k \)-module. Let \( V_0 \) be the underlying \( k \)-module. Then there is an injective \( G \)-morphism \( \rho : V \to V_0 \otimes A \).

**Proof.** It is easy to check that the coaction \( V_0 \otimes \Delta \) of \( V_0 \otimes A \) commutes with the comultiplication \( \Delta \), hence \( \rho \) is a homomorphism. The injectivity follows from the fact that \( (\text{id}_V \otimes \epsilon) \circ \rho \) is injective.

2.3 Proof

**Proof of theorem 8.** By combining (b) and lemma 10 we can extend our family \( (\alpha_V) \) to range over all representations \( V \) instead of only the finitely generated.
Let $A' = A \otimes R$, and let $\alpha' = \alpha_{A'}$ be the $R$-linear map belong the the regular representation $r$ of $G$ on $A'$. The multiplication $m : A' \otimes A' \to A'$ is a $G$-morphism for the representations $r \otimes r$ and $r$, because for all $x \in G(R)$ and $f \otimes f' \in A' \otimes A'$ we have

\begin{align*}
(r(g) \circ m)(f \otimes f')(x) &= (r(g)(f \cdot f'))(x) = (f \cdot f')(xg) \\
(m \circ (r(g) \otimes r(g)))(f \otimes f')(x) &= ((r(g)f) \cdot (r(g)f'))(x) = f(xg) \cdot f'(xg).
\end{align*}

By (a) and (b) we then get that $m \circ \alpha' = (\alpha' \otimes \alpha') \circ m$, i.e. that $\alpha'$ is a $k$-algebra morphism. Similarly $\Delta : A' \to A' \times A'$ is a $G$-morphism for the representation $r$ and $1 \otimes r$. Hence, by (a) and (b) we get $\Delta \circ \alpha' = (1 \otimes \alpha') \circ \Delta$. Now we may apply lemma 9 to $G_R$ to conclude that $\alpha' = r_A(g)$ for some $g \in G(R)$.

Now, we will prove that this $g$ is indeed the $G$ we are looking for. Let $r_V : G \to \text{Aut}(V)$ be a representation of $G$ that is finitely generated as $k$-module. Let $V_0$ be the underlying $k$-module. Then by lemma 11 we have an injective map $\rho : V \to V_0 \otimes A$. By definition of $g$ we know that $\alpha$ and $r(g)$ agree on $A$ and they agree on $V_0$ by (c). By (a) they then agree on $V_0 \otimes A$ and by (b) they agree on $V$, which is what we wanted to proof.

The existence of $g$ is proven. The uniqueness can be deduced by noticing that the regular representation is faithful.

References
